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Poincaré–Birkhoff–Witt property for bicovariant differential algebras on simple quantum groups

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Abstract. We investigate the possibility of constructing bicovariant differential calculi on quantum groups $SO_q(N)$ and $Sp_q(N)$ as a quantization of an underlying bicovariant bracket. We show that, in contrast to the $GL(N)$ and $SL(N)$ cases, neither of the possible graded SO and Sp bicovariant brackets (associated with a quasitriangular r-matrices) obey the Jacobi identity when the differential forms are Lie algebra-valued. The absence of a classical Poisson structure gives an indication that differential algebras describing bicovariant differential calculi on quantum orthogonal and symplectic groups are not of Poincaré–Birkhoff–Witt type.

The bicovariant differential calculus (BDC) for quantum groups initiated by Woronowicz's [1] provides a meaningful example of noncommutative differential geometry [2]. On the other hand, it also serves as the starting point for formulating a new class of gauge theories with a simple quantum group playing the role of a gauge group [3, 4]. Many of the phenomena, which one can encounter studying these theories have their origin in the theory of BDC. Thus, it is extremely important to investigate the general properties of BDC for simple quantum groups.

In this paper we aim to find whether external algebras on quantum groups $SO_q(N)$ and $Sp_q(N)$ are of Poincaré–Birkhoff–Witt (PBW) type, i.e. whether they possess a unique basis of lexicographically ordered monomials. This is not an academic question, since it has a strong influence on all differential geometry associated with these groups. In particular, if the PBW property is absent under quantization the classical system and its corresponding quantum system will have a different number of observables.

Our consideration is based on the R-matrix approach of [5], which is very useful in dealing with BDCs.

Recall that the central point of Woronowicz's theory is the construction of bicovariant bimodules Γ over a Hopf algebra $\mathcal{A}$ (the algebra of functions on a quantum group). The bimodules over $\mathcal{A}$ supplied with two coactions:

$$\Delta_R : \Gamma \to \Gamma \otimes \mathcal{A} \qquad \text{and} \qquad \Delta_L : \Gamma \to \mathcal{A} \otimes \Gamma \tag{1}$$

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satisfying the set of axioms [1]. Bicovariant bimodules are interpreted as noncommutative of tensor bundles Γ_{cl} over Lie groups. For the case of simple quantum groups the classification of bicovariant bimodules was obtained in [6] and confirmed in [7].

A first-order differential calculus is defined as a pair (Γ, d), where differential $\mathrm{d}: \mathcal{A} \to \Gamma$ is a nilpotent mapping obeying the Leibnitz rule.

The bicovariant wedge product of two left-invariant 1-forms Ω_i is defined via tensor algebra construction:

$$\Omega_i \wedge \Omega_j = \Omega_i \otimes_{\mathcal{A}} \Omega_j - \sigma_{ij}^{lk} \Omega_l \otimes_{\mathcal{A}} \Omega_k \tag{2}$$

or, in concise matrix notation,

$$\Omega_1 \wedge \Omega_2 = (I_{12} - \sigma_{12}) \Omega_1 \otimes_{\mathcal{A}} \Omega_2. \tag{3}$$

Here matrix σ_{12} satisfies the Yang–Baxter equation (YBE) [1]:

$$\sigma_{23}\sigma_{12}\sigma_{23} = \sigma_{12}\sigma_{23}\sigma_{12} \tag{4}$$

which provides the associativity of the wedge product:

$$(\Omega_1 \wedge \Omega_2) \wedge \Omega_3 = \Omega_1 \wedge (\Omega_2 \wedge \Omega_3). \tag{5}$$

Therefore, adopting (2), (3) one can construct, starting from Γ, an associative external algebra $\Gamma^{\wedge} = \sum_n \Gamma^{(n)}$, where $\Gamma^{(0)} = \mathcal{A}$, $\Gamma^{(1)} = \Gamma$ and $\Gamma^{(n)}$ is the space of n-forms. It is proved [1] that a first-order differential calculus can be lifted to higher-order differential forms via extending Γ by an additional bi-invariant 1-form X generating d: $\mathrm{d}\Omega = [X, \Omega]_{\pm}$, $\Omega \in \Gamma^{\wedge}$.

However, let us stress that YBE (4) does not guarantee that $\Gamma^{\wedge}$ constructed in such a way is a PBW-type algebra. From the point of view of general BDC theory [1] the fulfilment of the PBW property for quantum external algebras is an additional physical requirement.

The general properties of σ for quantum simple Lie groups (in particular, the projector expansion) were studied in [8, 9]. In [8] the authors modified definition (2) by imposing the additional quadratic relations on the generators Ω_i (we will comment on this later). The direct investigation of the PBW property for (2) is rather involved and, to our knowledge, it has been tackled only for the external algebras on $GL_q(N)$ and $SL_q(N)$ [10–12]. Fortunately, for quantum groups $SO_q(N)$ and $Sp_q(N) (N = 2n)$, one can exploit the ideas in [14, 15] that the quantum groups can be obtained by quantizing classical Poisson structures and trying to answer the question about the PBW property, on the semiclassical level only. This is due to the existence of the infinitesimal version of bicovariant differential calculi on the quantum groups provided by graded bicovariant brackets [16] which can be deduced directly as a semiclassical limit of the PBW algebras presented in [11–13]. In this approach, it is assumed that the anticommutator in Γ is determined, in the semiclassical approximation, by a graded bracket on Γ_{cl}:

$$\Omega \wedge \Omega' + \Omega' \wedge \Omega = \hbar\{\Omega, \Omega'\} + \hbar^2(\ldots) \tag{6}$$

satisfying the condition of bicovariance:

$$\Delta_{L,R}(\{\Omega, \Omega'\}) = \{\Delta_{L,R}(\Omega), \Delta_{L,R}(\Omega')\} \qquad \Omega, \Omega' \in \Gamma^1_{cl}. \tag{7}$$

The coactions $\Delta_L : \Gamma \to \mathcal{A} \otimes \Gamma$ and $\Delta_R : \Gamma \to \Gamma \otimes \mathcal{A}$ coming in (7) are the classical analogues of (1). On the matrix elements $\Omega_i^j (i, j = 1, 2, \ldots, N)$ of the Lie-valued left-invariant Cartan's form Ω (Ω_j^i generates $\Gamma_{cl}^{\wedge}$) $\Delta_{L,R}$ are defined as

$$\Delta_L(\Omega_j^i) = I \otimes \Omega_j^i \equiv \Omega_j^i \qquad \Delta_R(\Omega_j^i) = \Omega_l^k \otimes (T^{-1})_k^i T_j^l \equiv (T^{-1}\Omega T)_j^i \quad (8)$$

and, to an arbitrary element of $\Gamma_{cl}^{\wedge}$, they are extended as homomorphisms with respect to the wedge product. We introduce the matrix element T_j^i of the groups in (8) and, for the sake of simplicity, we omit the signs of the tensor products in the last parts of (8). In the following we will refer to the bracket satisfying (7) as bicovariant with respect to the L, R-coactions of G.

To turn $\Gamma_{cl}^{\wedge}$ into a classical Poisson system we also impose on the graded bracket in (6) the following conventional requirements:

(i) the symmetry condition:

$$\{\rho, \rho'\} = (-1)^{\deg(\rho)\deg(\rho')+1}\{\rho', \rho\}$$

(ii) the graded Jacobi identity:

$$(-1)^{\deg\rho_1 \deg\rho_3}\{\{\rho_1, \rho_2\}, \rho_3\} + (-1)^{\deg\rho_2 \deg\rho_3}\{\{\rho_3, \rho_1\}, \rho_2\} + (-1)^{\deg\rho_1 \deg\rho_3}\{\{\rho_2, \rho_3\}, \rho_1\} = 0 \quad (9)$$

and

(iii)

$$\{\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4\} = (-1)^{\deg\rho_2 \cdot \rho_3}\{\rho_1, \rho_3\} \otimes \rho_2\rho_4 + (-1)^{\deg\rho_2 \cdot \rho_1}\rho_1\rho_3 \otimes \{\rho_2, \rho_4\}.$$

In addition we demand (as usual) that this bracket be a graded differentiation:

$$\{\rho_1, \rho_2 \wedge \rho_3\} = \{\rho_1, \rho_2\} \wedge \rho_3 + (-1)^{\deg\rho_1 \cdot \deg\rho_2}\rho_2 \wedge \{\rho_1, \rho_3\}.$$

If a bracket satisfying the above requirements exists, then $\Gamma_{cl}^{\wedge}$ is said to be equipped with a graded Poisson–Lie (PL) structure [16] and one can consider $\Gamma_{cl}^{\wedge}$ as a phase space for some graded dynamical system.

Thus, in general, an algebra of quantum external forms is expected to be a graded bicovariant algebra with the graded commutator that produces, in the semiclassical limit, a graded bicovariant bracket. Now the fact that this algebra is of PBW type leads, in semiclassical theory, to the requirement for the corresponding bicovariant bracket to be Poisson, i.e. to satisfy the Jacobi identity (here and below we confine ourselves only to a consideration of the exterior algebras (2) having the usual classical limit).

In the cases of $GL(N)$ and $SL(N)$, graded PL structures exist [16, 17] and the corresponding algebras of quantum external forms are of PBW type [11, 12]. Moreover, if a graded PL structure exists, then bicovariance and the PBW property can be considered as main quantization principles.

Let G be $SO(N)$ or $Sp(N)$ groups and $\mathcal{G}$ be the corresponding Lie algebra. The following terminology (see [18]) will be useful. A skewsymmetric solution $r (r \in \mathcal{G} \wedge \mathcal{G})$ of the classical YBE (cYBE) will be referred to as a triangular r-matrix and a skewsymmetric r obeying the modified YBE (mYBE) will be referred to as a quasitriangular one.

Our strategy is as follows. It is natural to consider a graded PL structure on $\Gamma^{\wedge}_{cl}$ generated by the brackets between the components of $\mathcal{G}$-valued Cartan's form Ω, i.e. when $\Omega^i_j = \omega_\alpha(t^\alpha)^i_j$ where $t^\alpha \in \mathcal{G}$. It is shown below (see also [19]) that these brackets are defined via a triangular r-matrix. However, this is not for the case of the standard r-matrix [18] associated with simple Lie algebras. In other words, if we employ a quasitriangular r (that is relevant for subsequent quantization) and require the G-covariant Poisson bracket $\{\Omega_1, \Omega_2\}$ to be an element of $\mathcal{G}\otimes\mathcal{G}$ (as a matrix), then we get a unique solution $\{\Omega_1, \Omega_2\} = 0$ (see below). One may hope that by discarding the requirement $\Omega \in \mathcal{G}$ it would be possible to obtain a graded bicovariant bracket with a quasitriangular r. Below we analyse this possibility and, hence, assume the general situation when $\Omega \in \mathrm{Mat}(N, \mathbb{C}) \sim gl(N, \mathbb{C})$. In this case, a covariancy group G of graded brackets on $gl(N)$ is a subgroup of $GL(N)$. Let us stress that this is in agreement with the quantum external algebra construction [1] where $\dim \Gamma^{(1)} = N^2$. One remark is in order. We will not consider in this paper the graded bicovariant brackets which are covariant under the groups isomorphic with the linear groups of A_{n-1} series (e.g. $SO(3) \sim Sp(2) \sim SL(2)$, see [19] for discussion).

Now we recall briefly the basic facts about Lie groups G corresponding to $so(N)$ or $sp(N) (N = 2n)$ Lie algebras. The fundamental representation of G is given by

$$TCT^tC^{-1} = CT^tC^{-1}T = I$$

where $N \times N$ matrix C is $C^{ij} = \delta^{ij'}$ for $SO(N)$ and $C^{ij} = \epsilon_i\delta^{ij'}$ for $Sp(N)$, $i' = N + 1 - i$, $\epsilon_i = 1 (i = 1, \ldots, n)$, and $\epsilon_i = -1$ if $(i = n, \ldots, 2n)$. We denote by $C^{ij}(C_{ij})$ the matrix elements of $C(C^{-1})$.

The fundamental representations of the corresponding Lie algebras are defined as follows

$$\mathcal{G} = \{X \in \mathrm{Mat}(N, \mathbb{C}) | X^t = -CXC^{-1}\}.$$

To simplify the calculations we introduce an operation $\tilde{}$ acting on Ω, Ω^2, etc, in the following way

$$\tilde{\Omega} = C\Omega^tC^{-1} \qquad \widetilde{\Omega^2} = C(\Omega^2)^tC^{-1} = -(\tilde{\Omega})^2. \tag{10}$$

Clearly, $\tilde{\tilde{\Omega}} = \Omega$. Using this operation we split matrix-valued forms as $\Omega^{\pm} = \Omega \pm \tilde{\Omega}$. Note that the form Ω^- belongs to $\mathcal{G}$ in the fundamental representation.

It can be shown [20] that the general form of a Z-graded bicovariant bracket $\{\Omega_1, \Omega_2\}$ is

$$\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]]_+ + \mathrm{Tr}_{34}(W_{1234}\Omega_3\Omega_4) \tag{11}$$

where r_{12} is the quasitriangular r-matrix and W_{1234} is a G-invariant tensor:

$$W_{1234} = T_1T_2T_3T_4W_{1234}T_1^{-1}T_2^{-1}T_3^{-1}T_4^{-1} \tag{12}$$

with symmetry properties:

$$W_{1234} = W_{2134} = -W_{1243}. \tag{13}$$

Here indices 1, 2, 3, 4 denote the numbers of the matrix spaces. Thus, to construct a general SO- (Sp)-bicovariant bracket we have to enumerate all tensors (13) that are invariant under

G-action (12). Classification of all possible W_{1234} leads to the following explicit form of bracket (11) (the detailed proof of this statement will be published elsewhere):

$$\begin{aligned}\{\Omega_1, \Omega_2\} = [\Omega_1[\Omega_2, r_{12}]]_+ + X_{12}^{(1)}(\Omega_1^2 + \Omega_2^2) + (\tilde{\Omega}_1^2 + \tilde{\Omega}_2^2)X_{12}^{(2)} + (\tilde{\Omega}_1 X_{12}^{(3)}\Omega_1 + \tilde{\Omega}_2 X_{12}^{(3)}\Omega_2) \\ + (\tilde{\Omega}_1 X_{12}^{(4)}\Omega_2 + \tilde{\Omega}_2 X_{12}^{(4)}\Omega_1) + X_{12}^{(5)}(\Omega_1\tilde{\Omega}_1 + \Omega_2\tilde{\Omega}_2) \\ + (X_{12}^{(6)}(\tilde{\Omega}_1 + \tilde{\Omega}_2) + (\Omega_1 + \Omega_2)X_{12}^{(7)}) \operatorname{tr}\Omega\end{aligned} \tag{14}$$

where all $X^{(i)}$ are symmetric G-invariant matrices in $\mathrm{Mat}(N, \mathbb{C}) \times \mathrm{Mat}(N, \mathbb{C})$:

$$X^{(i)} = a_i I + b_i P + c_i K^0$$

and a_i, b_i, c_i are complex numbers, I is the identity matrix, P is a permutation matrix and K^0: $(K^0)_{ij}^{kl} = C^{kl}C_{ij}$.

Due to the identities

$$K_{12}^0\Omega_1 = K_{12}^0\tilde{\Omega}_2 \qquad K_{12}^0\Omega_2 = K_{12}^0\tilde{\Omega}_1 \tag{15}$$

we find that $K_{12}^0(\Omega_1\tilde{\Omega}_1 + \Omega_2\tilde{\Omega}_2) = 0$, i.e. we can put $c_5 = 0$ and, therefore, the bracket (14) depends on 20 arbitrary parameters a_i, b_i, c_i. In fact this number coincides with the dimension of the cohomology group $H^0(\mathcal{G}, SV \otimes \wedge V)$, where $V = \mathrm{Mat}(N, \mathbb{C})$ and $SV(\wedge V)$ stands for the symmetric (antisymmetric) part of $V \otimes V$. We note that operators $X^{(i)}$ have the matrix structure of Yangian R-matrices.

Having the general form (14) one can calculate the bracket between the variables $\Omega^\pm$. For this purpose one needs explicit expressions for $\{\tilde{\Omega}_1, \Omega_2\}$ and $\{\Omega_1, \tilde{\Omega}_2\}$ that are obtained from (14) by acting with $\tilde{}$ in the first or second matrix spaces. Now if we take into account that

$$\widetilde{X_{12}^{(i)}}(a, b, c) = X_{12}^{(i)}(a, \epsilon c, \epsilon b)$$

then we get

$$\begin{aligned}\{\Omega_1^\pm, \Omega_2^\pm\} = [\Omega_1^\mp[\Omega_2^\pm, r_{12}]]_+ + Z_{12}^\pm(\Omega_1^2 + \Omega_2^2) - (\tilde{\Omega}_1^2 + \tilde{\Omega}_2^2)Z_{12}^\pm \\ + (V_{12}^\pm(\Omega_1 + \Omega_2) + (\tilde{\Omega}_1 + \tilde{\Omega}_2)V_{12}^\pm) \operatorname{tr}\Omega\end{aligned} \tag{16}$$

where

$$\begin{aligned} Z_{12}^\pm &= (X_{12}^1 - X_{12}^2) \pm (\tilde{X}^1 - \tilde{X}^2) = \alpha^\pm Y_{12}^\pm + 2\delta_{\pm +}(a_1 - a_2) \\ V_{12}^\pm &= (X_{12}^6 + X_{12}^7) \pm (\tilde{X}^6 + \tilde{X}^7) = \beta^\pm Y_{12}^\pm + 2\delta_{\pm +}(a_6 + a_7) \\ \alpha^\pm &= b_1 - b_2 \pm \epsilon(c_1 - c_2) \\ \beta^\pm &= b_6 + b_7 \pm \epsilon(c_6 + c_7) \end{aligned} \tag{17}$$

and $Y_{12}^\pm = P_{12} \pm \epsilon K_{12}^0$. Thus, we see that Lie-valued generators Ω^- form the closed algebra

$$\{\Omega_1^-, \Omega_2^-\} = [\Omega_1^-[\Omega_2^-, r_{12}]]_+ \tag{18}$$

only if $Z_{12}^- = V_{12}^- = 0$ or $\alpha^- = \beta^- = 0$. Then the calculation of the Jacobi identity (9) gives

$$\{\{\Omega_1^-, \Omega_2^-\}, \Omega_3^-\} + (\text{cycle } 1, 2, 3) = -[\Omega_1^-, [\Omega_2^-, [\Omega_3^-, C(r)]]_+] \tag{19}$$

where

$$C(r) = [r_{12}, r_{23} + r_{13}] + [r_{13}, r_{23}]. \tag{20}$$

If r_{12} is a quasitriangular r-matrix ($C(r) \neq 0$ is ad-invariant tensor), i.e. r_{12} is a solution of the mYBE (20), then the related bracket (18) is non-Poisson (see (19)). Correspondingly, if r_{12} is a triangular r-matrix ($C(r) = 0$ in equation (20)), then the bracket (18) is Poisson. These statements agree with the results of [19].

Before considering the general bracket (14) we recall how the exterior derivative d comes into this scheme. If we relate in the quantum case the co-invariant element X of Woronowicz with the quantum trace $\mathrm{tr}_q\,\Omega$ (the definition of the q-trace see in [5, 21, 13, 4, 22]), then semiclassically it means that the ordinary exterior derivative d is expressed via the corresponding bicovariant bracket:

$$\mathrm{d} = \frac{1}{\kappa}\{\mathrm{tr}\,\Omega, \ldots\} \tag{21}$$

where κ is some numerical parameter depending on a bracket under consideration. The fulfilment of the nilpotency condition: $\mathrm{d}^2 = 0$ is equivalent to the identity:

$$\{\{\Omega, \mathrm{tr}\,\Omega\}, \mathrm{tr}\,\Omega\} = 0 \tag{22}$$

and the Leibnitz rule is guaranteed by

$$\{\{\Omega_1, \Omega_2\}, \mathrm{tr}\,\Omega\} + \{\{\Omega_1, \mathrm{tr}\,\Omega\}, \Omega_2\} - \{\Omega_1, \{\Omega_2, \mathrm{tr}\,\Omega\}\} = 0. \tag{23}$$

Bracket (14) satisfying (22) and (23) will be referred to as differential. If (14) satisfy the Jacobi identity (9) then (22) and (23) are fulfilled automatically. It is worth noting that $\mathrm{tr}\,\Omega\,((\mathrm{tr}\,\Omega)^2 = 0)$ looks like a BRST charge.

First, we find all differential brackets. From (14) one can extract the general forms

$$\{\Omega, \mathrm{tr}\,\Omega\} = \mu_1\Omega^2 + \mu_2\tilde{\Omega}^2 + \mu_3\tilde{\Omega}\Omega + \mu_4\Omega\tilde{\Omega} + (\mu_5\tilde{\Omega} + \mu_6\Omega)\,\mathrm{tr}\,\Omega \tag{24}$$

and

$$\{\tilde{\Omega}, \mathrm{tr}\,\Omega\} = -\mu_1\tilde{\Omega}^2 - \mu_2\Omega^2 - \mu_3\tilde{\Omega}\Omega - \mu_4\Omega\tilde{\Omega} + (\mu_5\Omega + \mu_6\tilde{\Omega})\,\mathrm{tr}\,\Omega \tag{25}$$

where

$$\begin{aligned}
\mu_1 &= 2b_1 + \epsilon c_1 - \epsilon c_2 + \epsilon c_4 + Na_1 \\
\mu_2 &= -\epsilon c_1 + \epsilon c_2 + \epsilon c_4 + 2b_2 + Na_2 \\
\mu_3 &= \epsilon c_3 + b_3 + 2b_4 + Na_3 \\
\mu_4 &= \epsilon c_3 - b_3 + 2b_5 + Na_5 \\
\mu_5 &= a_4 + Na_6 + 2b_6 + \epsilon(c_6 + c_7) \\
\mu_6 &= -a_4 + Na_7 + 2b_7 + \epsilon(c_6 + c_7).
\end{aligned} \tag{26}$$

Substitution of (26) in (22) gives four solutions for coefficients μ:

(i) $\mu_1 = \mu$, $\mu_2 = \mu_3 = \mu_4 = \nu$, $\mu_5 = \mu_6 = 0$, where μ, ν are arbitrary numbers except $\mu = \nu = 0$;

(ii) $\mu_1 = \mu_2 = -\mu_3 = -\mu_4 \neq 0$, $\mu_5 = \mu_6 = 0$;

(iii) $\mu_5 = -\mu_6 = a_4 \neq 0$, $\mu_i = 0$ $(i = 1, \ldots, 4)$;

(iv) $\mu_i = 0$ $(i = 1, \ldots, 6)$.

Thus, for bracket (24) (for (25) respectively) we have four possibilities:

$$\begin{aligned}
&\text{(i) } \{\Omega, \operatorname{tr}\Omega\} = \mu\Omega^2 + \nu(\tilde{\Omega}^2 + \tilde{\Omega}\Omega + \Omega\tilde{\Omega}) \qquad (\text{for all } \mu, \nu \text{ except } \mu = \nu = 0),\\
&\text{(ii) } \{\Omega, \operatorname{tr}\Omega\} = \mu(\Omega^2 + \tilde{\Omega}^2 - \tilde{\Omega}\Omega - \Omega\tilde{\Omega}) \qquad (\mu \neq 0),\\
&\text{(iii) } \{\Omega, \operatorname{tr}\Omega\} = \mu(\tilde{\Omega} - \Omega)\operatorname{tr}\Omega \qquad (\mu = a_4 \neq 0),\\
&\text{(iv) } \{\Omega, \operatorname{tr}\Omega\} = 0.
\end{aligned} \tag{27}$$

The next step is to impose identity (23) proving the Leibnitz rule for the differential d (21). This was done by using the symbolic manipulation program REDUCE. The resulting differential bicovariant brackets are presented in the appendix.

Now substituting the calculated coefficients in (14) and analysing the identity (9) with the help of the REDUCE program, we arrive at the conclusion that neither of the nontrivial differential brackets is Poisson. Thus, among the family (14) of bicovariant brackets there are differential brackets but no Poisson brackets. Note that we essentially use the requirement that the Ω_s lie in the algebras: (1) $so(N)$, $sp(2n)$ or in (2) $gl(N) = \mathrm{Mat}(N)$. In the first case we have an additional relation on the generators $\Omega^+ = 0$. We stress that if we consider some other relations (cubic relations or $\Omega_1\Omega_2 K_{12}^0 = -K_{12}^0\Omega_1\Omega_2$), then the Poisson structure can exist.

Now we analyse the external bicovariant algebra (3) on quantum groups $SO_q(N)$ and $Sp_q(N)$ directly in quantum case. For these q-groups the R-matrix satisfies the cubic characteristic equation [5]:

$$R = R^{-1} + \lambda - \lambda K \qquad K \equiv -\frac{1}{\lambda\nu}(R^2 - \lambda R - 1) \tag{28}$$

where $\nu = \epsilon q^{\epsilon - N}$, $\lambda = q - q^{-1}$, $R = \hat{R}_{12} = P_{12}R_{12}$ and the matrix $K = K_{12} = K^{i_1 i_2}_{j_1 j_2} = C^{i_1 i_2} C_{j_1 j_2}$ is proportional to the singlet projector $P^{(0)}$:

$$P^{(0)} = \mu^{-1} K \qquad \mu = (1 + \epsilon[N - \epsilon]_q). \tag{29}$$

Note that this time C is a quantum matrix [5]. Below we also use the projectors:

$$P^{(\pm)} = \frac{1}{q + q^{-1}}(\pm R + q^{\mp 1} I + \mu_\pm K) \qquad \mu_\pm = -\frac{q^{\mp 1} \pm \nu}{\mu}. \tag{30}$$

It has been shown in [8, 9] that for differential 1-forms one has the following relations coming from definition (3):

$$X^{(\pm\pm)} = P^{(\pm)}\Omega' R \Omega' P^{(\pm)} = 0 \qquad X^{(00)} = P^{(0)}\Omega' R \Omega' P^{(0)} = 0. \tag{31}$$

Here $\Omega' = I \otimes \Omega = \Omega_2$ and the signs of the wedge products are omitted. Taking the following sum

$$qX^{(++)} + \frac{1}{q}X^{(--)} - \frac{q\mu_+^2 + q^{-1}\mu_-^2}{(q + q^{-1})^2} X^{(00)} = 0 \tag{32}$$

and using the identities

$$\frac{\mu_+ + \mu_-}{q+q^{-1}} = -\frac{1}{\mu} \qquad \frac{q\mu_+ - q^{-1}\mu_-}{q+q^{-1}} = -\frac{\nu}{\mu} \tag{33}$$

one can show that relations (31) are equivalent to the unique relation:

$$(R\Omega' R\Omega' R + \Omega' R\Omega') - \frac{1}{\mu}(K\Omega' R\Omega' + \Omega' R\Omega' K) - \frac{\nu}{\mu}(K\Omega' R\Omega' R + R\Omega' R\Omega' K) = 0. \tag{34}$$

This form for the defining relations for $\Gamma^\wedge$ is suitable for producing a graded bicovariant bracket on $\Gamma^\wedge_{cl}$.

The semiclassical expansions of projector $P^{(0)}_{12}$ and R-matrix are:

$$P^{(0)}_{12} = \hat{P}^{(0)}_{12} + \hbar\frac{\epsilon}{N}K^1_{12} + \mathrm{O}(\hbar^2) \qquad R_{12} = P_{12} + \hbar P_{12}\tilde{r}_{12} + \mathrm{O}(\hbar^2)$$

where $\hat{P}^{(0)}_{12} = (\epsilon/N)K^0_{12}$ and $\tilde{r}$ satisfies CYBE. It follows from $KR = RK = \nu K$ that in the first order in $\hbar$:

$$\begin{aligned} K^1_{12} - \epsilon K^1_{12}P_{12} &= K^0_{12}\tilde{r}_{12} - \epsilon(1-\epsilon N)K^0_{12} \\ K^1_{12} - \epsilon P_{12}K^1_{12} &= \tilde{r}_{21}K^0_{12} - \epsilon(1-\epsilon N)K^0_{12}. \end{aligned} \tag{35}$$

Then, by expanding (34) in powers of $\hbar$, taking into account (35) and the correspondence (6), we obtain

$$(I - \hat{P}^{(0)}_{12})(\{\Omega_1, \Omega_2\} + G_{12}) - (\{\Omega_1, \Omega_2\} + G_{12})\hat{P}^{(0)}_{12} = 0 \tag{36}$$

where

$$G_{12} = -[\Omega_1, [\Omega_2, r_{12}]]_+ + P_{12}(\Omega_1^2 + \Omega_2^2) - \epsilon(K_{12}\Omega_1\Omega_2 + \Omega_1\Omega_2K_{12} + \Omega_1K_{12}\Omega_2 + \Omega_2K_{12}\Omega_1) \tag{37}$$

and we made use of the quasitriangular r-matrix: $r = \tilde{r} - (P - \epsilon K)$.

The components $\hat{P}^{(0)}_{12}\{\Omega_1, \Omega_2\}(I - \hat{P}^{(0)}_{12})$ are not defined by (36). Thus, we see that relations (31) are insufficient to generate, in the limit $\hbar \to 0$, a genuine bicovariant bracket. In the quantum case it means that the number of defining relations (31) is not enough to reorder the lexicographically arbitrary monomial in Ω_s. Therefore, if we confine ourselves only to (31), then we cannot conclude that $\dim \Gamma^\wedge$ is equal to $\dim \Gamma^\wedge_{cl}$.

On the other hand, we cannot assume the solution of (36) to be

$$\{\Omega_1, \Omega_2\} = -G_{12} \tag{38}$$

since G_{12} is symmetric under $1 \leftrightarrow 2$ only if the following relation holds:

$$K_{12}\Omega_1\Omega_2 + \Omega_1\Omega_2K_{12} = 0. \tag{39}$$

But this relation contradicts the requirement that $\Omega \in \mathcal{G}$ or that the number of Ω_s are $N^2(\Omega \in \mathrm{Mat}(N))$. Note, however, that the bracket (38) is Poissonian for Ω_s restricted by constraint (39).

To improve the situation the authors of [8], in addition to (31), have assumed the relations (in the $SO_q(N)$-case):

$$X^{(0+)} = P^{(0)}\Omega' R\Omega' P^{(+)} = 0 \qquad X^{(+0)} = P^{(+)}\Omega' R\Omega' P^{(0)} = 0. \tag{40}$$

One can obtain without problems that (34) and (40) are equivalent to the relation:

$$R\Omega' R\Omega' R + \Omega' R\Omega' + \frac{1}{\mu}(\nu q^{-1} - 1)(K\Omega' R\Omega' + \Omega' R\Omega' K) = 0. \tag{41}$$

By expanding (41) in $\hbar$, as was done for the general relation (34), we get the following bicovariant bracket:

$$\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]]_+ - P_{12}(\Omega_1^2 + \Omega_2^2) + (\Omega_1 K_{12}\Omega_2 + \Omega_2 K_{12}\Omega_1). \tag{42}$$

as a particular case of (14). Now we see that, according to our classification, this bracket is neither Poissonian nor differential. This means in the quantum case that the requirement $\mathrm{d}^2 = 0$ (22) implies some additional cubic relations on generators Ω^i_j, which were not assumed from the beginning. The situation is somewhat improved when we require $\mathrm{d}^2 = 0$ only on the 'physical' components $\Omega = \Omega^-$. This requirement is consistent with (22), since $\{\Omega^-, \mathrm{tr}\,\Omega\} = -2(\Omega^-)^2$. However, the substitution $\Omega \to \Omega^-$ in (23) leads to the conclusion:

$$\{\{\Omega_1^-, \Omega_2^-\}, \mathrm{tr}\,\Omega\} + \{\{\Omega_1^-, \mathrm{tr}\,\Omega\}, \Omega_2^-\} - \{\Omega_1^-, \{\Omega_2^-, \mathrm{tr}\,\Omega\}\} \neq 0.$$

Thus, one cannot assume the Leibnitz rule for d on the 'physical' subalgebra generated by Ω^- without imposing new cubic relations on Ωs. Note that if we impose the unacceptable relations (39), then bracket (42) coincides with (38) and, therefore, is Poissonian.

Seemingly, the absence of a bicovariant Poisson structure for $SO(N)$ and $Sp(N)$ (N is generic) reflects the fact that we cannot confine ourselves by considering only G-invariant tensors W in (11). Considering in (11) tensor W which is not G-invariant, we disturb the bicovariance but may hope to keep the Jacobi identity. Then we expect that the bicovariance will be restored on the surface $\Omega^+ = 0$ if we treat $\Omega^+ = 0$ as the first-order constraint (in the Dirac sense).

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Appendix. Differential SO- and sp-covariant brackets on Mat(N)

First solution: $\{\Omega, \mathrm{tr}\,\Omega\} = \mu\Omega^2 + \nu(\tilde{\Omega}^2 + \tilde{\Omega}\Omega + \Omega\tilde{\Omega})$

(i)

$$\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]]_+ - \frac{1}{N}(2b_2 + \epsilon c_3 + Na_2)(\Omega_1^2 + \Omega_2^2) + a_2((\Omega_1^+)^2 + (\Omega_2^+)^2)$$
$$+ c_3\Omega_1^+ K_{12}\Omega_2^+ + \tfrac{1}{2}\mu P_{12}(\Omega_1^2 + \Omega_2^2) + b_2 P_{12}((\Omega_1^+)^2 + (\Omega_2^+)^2)$$
$$+ c_6(K_{12}\Omega_1^+ - \Omega_1^+ K_{12})\operatorname{tr}\Omega$$

(ii)

$$\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]]_+ - \frac{1}{N}(2b_1 + \epsilon c_3 + Na_2)(\Omega_1^2 + \Omega_2^2) + a_2((\Omega_1^+)^2 + (\Omega_2^+)^2)$$
$$+ c_3\Omega_1^+ K_{12}\Omega_2^+ + (2b_1 + \nu) P_{12}((\Omega_1^+)^2 + (\Omega_2^+)^2)$$
$$-(b_1+\nu)(P_{12}(\Omega_1^2+\Omega_2^2)+(\tilde\Omega_1^2+\tilde\Omega_2^2)P_{12}) - \tfrac{1}{2}\epsilon\nu(K_{12}(\Omega_1^2+\Omega_2^2)+(\Omega_1^2+\Omega_2^2)K_{12})$$
$$+ c_6(K_{12}\Omega_1^+ - \Omega_1^+ K_{12})\operatorname{tr}\Omega$$

(iii)

$$\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]]_+ - \frac{1}{N}(2b_2 + \epsilon c_3 - \nu)((\Omega_1^+)^2 + (\Omega_2^+)^2)$$
$$+ c_3\Omega_1^+ K_{12}\Omega_2^+ + b_1 P_{12}((\Omega_1^+)^2 + (\Omega_2^+)^2) + (a_6(\tilde\Omega_1 + \tilde\Omega_2) - 3a_6(\Omega_1 + \Omega_2)$$
$$- \tfrac{1}{2}\epsilon(c_6 + c_7) P_{12}(\Omega_1^+ + \Omega_2^+) + c_6 K_{12}\Omega_1^+ + c_7\Omega_1^+ K_{12})\operatorname{tr}\Omega.$$

Second solution: $\{\Omega, \operatorname{tr}\Omega\} = \mu(\Omega^2 + \tilde\Omega^2 - \tilde\Omega\Omega - \Omega\tilde\Omega)$.

$$\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]]_+ + a_1((\Omega_1^-)^2 + (\Omega_2^-)^2) + c_3\Omega_1^- K_{12}\Omega_2^-$$
$$- \tfrac{1}{2}b_3 P_{12}(\Omega_1^+\Omega_1^- + \Omega_2^+\Omega_2^-) + (-\epsilon c_1 P_{12} + c_1 K_{12})(\Omega_1^2 + \Omega_2^2)$$
$$+ (\tilde\Omega_1^2 + \tilde\Omega_2^2)(\epsilon c_1 P_{12} + c_2 K_{12})((a_6 + \epsilon c_6 P_{12} + c_6 K_{12})\Omega_1^+$$
$$+ \Omega_1^+(a_6 + \epsilon c_6 P_{12} + c_6 K_{12}))\operatorname{tr}\Omega$$

where $b_3 = -\epsilon(c_1 + c_2)$.
Third solution: $\{\Omega, \operatorname{tr}\Omega\} = \mu(\tilde\Omega - \Omega)\operatorname{tr}\Omega$.

$$\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]]_+ + (-a_3 - b_4 + c_1 K_{12})(\Omega_1^2 + \Omega_2^2)$$
$$+ (\tilde\Omega_1^2 + \tilde\Omega_2^2)(a_3 + b_4 + c_1 K_{12})(a_3 + b_4 P_{12})(\Omega_1^+\Omega_1^- + \Omega_2^+\Omega_2^-)$$
$$+ (\mu + b_3 P_{12})(\tilde\Omega_2\Omega_1 + \tilde\Omega_1\Omega_2) + (X_{12}^{(6)}(\tilde\Omega_1 + \tilde\Omega_2) + (\Omega_1 + \Omega_2)X_{12}^{(7)})\operatorname{tr}\Omega$$

where $b_3 = -(Na_3 + 2b_4)$ and coefficients in $X_{12}^{(6)}$ and in $X_{12}^{(7)}$ remain arbitrary.

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